§1 Modular group and its action on $\mathbb{H}$.

<u>Defn</u> For any commutative ring R, the <u>general linear group over R</u>

$GL_2(R)$ is defined as

$$GL_2(R) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a, b, c, d \in R \\ ad - bc \in R^\times \end{array} \right\}.$$

<u>Special linear group</u>

$$SL_2(R) := \{ g \in GL_2(R) \mid \det g = 1 \}$$

We will be mostly concerned with the cases $R = \mathbb{R}, \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$.

We <u>restrict</u> ourselves <u>first</u> to $GL_2(\mathbb{R}) = G$

G acts on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by Linear fractional transformations

$$G \times \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} = g, z\right) \longmapsto g \cdot z := \begin{cases} \dfrac{az+b}{cz+d} & \text{if } cz+d \neq 0 \\ \dfrac{a}{c} & \text{if } z = \infty,\ c \neq 0 \\ \infty & \text{if } z = \infty,\ c = 0 \\ \infty & \text{if } z = -\dfrac{d}{c} \end{cases}$$

To see that this is indeed a group action put $j(g,z) := cz+d$ $\quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

if $cz+d \neq 0$ (ie $z \neq -d/c$)

then $g\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} = (cz+d)\begin{pmatrix} g\circ z \\ 1 \end{pmatrix}$

$$= j(g,z)\begin{pmatrix} g\circ z \\ 1 \end{pmatrix}$$

For $h, g \in G$ calculating $gh\begin{pmatrix} z \\ 1 \end{pmatrix}$ in 2 different ways and using associativity of the matrix multiplication we get:

(I) $(gh)\begin{pmatrix} z \\ 1 \end{pmatrix} = j(gh, z)\begin{pmatrix} (gh)\circ z \\ 1 \end{pmatrix}$

(II) $g\left(h\begin{pmatrix} z \\ 1 \end{pmatrix}\right) = g\left(j(h,z)\begin{pmatrix} h\circ z \\ 1 \end{pmatrix}\right)$

$$= j(h,z)\, g\begin{pmatrix} h\circ z \\ 1 \end{pmatrix}$$

$$= j(h,z)\, j(g, h\circ z)\begin{pmatrix} g\circ(h\circ z) \\ 1 \end{pmatrix}$$

Second rows of (I) and (II) gives

$$\boxed{j(gh,z) = j(g, h\circ z)\, j(h,z)}$$

(automorphy condition
cocycle condition)

and now the 1st row gives

$$\boxed{g\circ(h\circ z) = (gh)\circ z}$$

ie this is indeed a group action

Note $I\circ z = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\circ z = z$.

A simple calculation gives

$$\boxed{\operatorname{Im}(g\circ z) = \det g \cdot \frac{\operatorname{Im} z}{|cz+d|^2}}$$

Hence if $\det g > 0$, in particular for $g \in SL_2(\mathbb{R})$
$g\circ z \in \mathbb{H}$ whenever $z \in \mathbb{H}$.
$g\circ z \in \mathbb{R}\cup\{\infty\}$ whenever $z \in \mathbb{R}\cup\{\infty\}$

$$SL_2(\mathbb{R}) \times \mathbb{H} \longrightarrow \mathbb{H},$$

Now we restrict ourselves to $SL_2(\mathbb{R})$ and $\mathbb{H}\cup\mathbb{R}\cup\{\infty\}$

Note this action is not faithful
(faithful means : $g\cdot x = x \Rightarrow g = e$)
$-I$ acts trivially. To make it faithful one can use instead $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$
the group of transformations defined by $SL_2(\mathbb{R})$

Rk Note for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$

The corresponding Möbius function

$$\Phi_M : \mathbb{C} \to \mathbb{C}$$
$$z \to Mz = (az+b)/cz+d$$

is a meromorphic function on $\mathbb{C}$

if $c=0$ then Φ_M has no poles

and if $c \neq 0$, then it has a simple pole at $z = -d/c$. Φ_M can be extended to the Riemann sphere $\mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$ by letting $M\infty := \begin{cases} \infty & \text{if } c=0 \\ \frac{a}{c} & \text{if } c \neq 0. \end{cases}$

We have the following properties

1. If $L, M \in GL_2(\mathbb{C})$ then $\Phi_L \circ \Phi_M = \Phi_{LM}$. In particular $\Phi_M : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a bijection, and $\Phi_M^{-1} = \Phi_{M^{-1}}$

2. $\Phi_M = Id \iff M = \lambda I$ for some $0 \neq \lambda \in \mathbb{C}$

3. If $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$, $z, w \in \mathbb{C} \setminus \{-d/c\}$ for $c \neq 0$, then $Mz - Mw = \det M \dfrac{z-w}{(cz+d)(cw+d)}$

4. If $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$, and $z \in \mathbb{C} \setminus \{-d/c\}$ for $c \neq 0$, then $\dfrac{d}{dz} Mz = \dfrac{\det M}{(cz+d)^2}$

5. If $M \in GL_2(\mathbb{R})$, then $\mathrm{Im}(Mz) = \det M \dfrac{\mathrm{Im}\, z}{|cz+d|^2}$ let $w = \bar{z}$ in (3)

6. Φ_M maps circles in $\hat{\mathbb{C}}$ to circles

⑦ If $U \subset \hat{\mathbb{C}}$ open

and $\operatorname{Aut} U := \{f : U \to U,\ f \text{ biholom}\}$
the Automorphism gp of U.

Then ① $\operatorname{Aut} \hat{\mathbb{C}} \cong PSL_2(\mathbb{C})$

② $\operatorname{Aut} \mathbb{C} = \{ z \mapsto az+b \mid a, b \in \mathbb{C},\ a \neq 0 \}$

③ $\operatorname{Aut} \mathbb{H} = PSL_2(\mathbb{R})$.

<u>Classification of LFTs</u>

We can classify linear F. T in $PSL_2(\mathbb{R})$ according to its fixed points.

Each $I \neq g \in PSL_2(\mathbb{R})$ has at most 2 fixed points in $\mathbb{C}$ and at least one fixed point

$$\frac{az+b}{cz+d} = z \Rightarrow cz^2 + (d-a)z - b = 0$$

<u>Thm 1</u> Let $\pm I \neq M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$.

Then M has either one or 2 fixed points in $\mathbb{R} \cup \{\infty\}$ or two complex conjugate (non-real) fixed points.

More precisely we have one of the following

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1) M is parabolic $\Leftrightarrow$ M has exactly one fixed point $p \in \mathbb{R} \cup \{\infty\}$

$\Leftrightarrow \operatorname{tr} M = \pm 2$

$\Leftrightarrow$ M is conjugate in $SL_2(\mathbb{R})$ to $\pm\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$

$\lambda \in \mathbb{R}$.

$\boxed{\begin{array}{l} z \to z + \lambda \\ \text{translation} \end{array}}$

2) M is elliptic $\Leftrightarrow$ M has exactly 2 complex conjugate (non-real) fixed points

$\Leftrightarrow |\operatorname{tr} M| < 2$

$\Leftrightarrow$ M is conjugate in $SL_2(\mathbb{R})$ to $\pm\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = k(\theta)$

$-\pi < \theta < \pi$, $\theta \neq 0$.

$\boxed{\begin{array}{l} z \to k(\theta) z \\ \text{rotation} \end{array}}$

3) M is hyperbolic $\Leftrightarrow$ M has 2 distinct fixed points in $\mathbb{R} \cup \{\infty\}$

$\Leftrightarrow |\operatorname{tr} M| > 2$

$\Leftrightarrow$ M is conjugate in $SL_2(\mathbb{R})$ to $\pm\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$

$\lambda \in \mathbb{R}$, $|\lambda| \neq 1$

$\boxed{\begin{array}{l} z \to \lambda^2 z \\ \text{dilation} \end{array}}$

(Proof See Polka (Intro to Complex function theory) Thm IX 2.8)

Next we look at different realization's of the upper Half plane.

(1) $\mathbb{H}$ as a homog. space

We start by noting that the action of $G = SL_2(\mathbb{R})$ on $\mathbb{H}$ is transitive i.e. There is only 1 orbit. In fact the subgroup $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL_2(\mathbb{R}) \right\}$ already acts transitively.

To see this for given $z = x + iy \in \mathbb{H}$ let $B = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix}$. Then

$$B \circ i = \frac{y^{1/2} i + xy^{-1/2}}{y^{-1/2}} = iy + x$$

Hence every $z \in \mathbb{H}$ is in the orbit of i

The stabilizer of $i = G_i := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \,\middle|\, g \circ i = i \right\}$

$$= \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \,\middle|\, a^2 + b^2 = 1 \right\} = SO_2(\mathbb{R})$$

$$\left(\frac{ai + b}{ci + d} = i \Rightarrow ai + b = di - c \Rightarrow \begin{array}{l} a = d \\ b = -c \end{array} \quad \det = a^2 + b^2 = 1 \right)$$

Recall from Algebra: If G is a group acting on a set X transitively and $x \in X$, then X can be identified w/ G/G_x

In particular $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \to \mathbb{H}$

$$gSO_2(\mathbb{R}) \to g\circ i$$

gives the identification of $\mathbb{H}$ with $SL_2(\mathbb{R})/SO_2(\mathbb{R})$. Note $SO_2(\mathbb{R})$ is a compact subgp of $SL_2(\mathbb{R})$

Rk. This way of viewing $\mathbb{H}$ leads naturally to generalizations.

Eg (1) $G = SL(2,\mathbb{C})$

$$K = SU(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SL_2(\mathbb{C}) \right\}$$

$$G/K \cong \{ P = z + rj \mid z \in \mathbb{C},\ r>0 \} = \mathbb{H}^3$$

(2) $G = Sp(n,\mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n,\mathbb{R}) \;\middle|\; {}^tMJM = J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \right\}$

$$K = \left\{ \begin{pmatrix} A & B \\ -B^t & A \end{pmatrix} \;\middle|\; A+Bi \text{ unitary} \in U(n,\mathbb{C}) \right\}$$

H_n = Siegel upper half plane

$$= \{ Z = X + iY \mid Z^t = Z,\ Y > 0 \}$$

$$\cong Sp(n,\mathbb{R})/K. \quad (Sp(1,\mathbb{R}) = SL(2,\mathbb{R}))$$

(3) $G = GL(n,\mathbb{R})$, $K = SO(n,\mathbb{R})$

$$G/K = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} y_2\cdots y_n & & & \\ & y_2\cdots y_{n-1} & & \\ & & \ddots & \\ & & y_2y_3 & \\ & & & y_1 \end{pmatrix} \;\middle|\; \begin{matrix} x \in \\ y_i > 0 \end{matrix} \right\}$$

$G = NAK$ A max'l abelian $(a_1 \cdots a_n)$, $N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ Nilpotent

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Recall any group G acts on G/K for a s/gp K by translation

$$G \times G/K \longrightarrow G/K$$
$$(g, hK) \longmapsto ghK.$$

Easy to see that

$$\begin{tikzcd} SL_2(\mathbb{R})/K \arrow[r, "g"] & SL_2(\mathbb{R})/K \\ hK \arrow[r, mapsto] \arrow[d, "\wr"] & ghK \arrow[d] \\ h\circ i \arrow[r, "g"] & (gh)\circ i \\ \mathbb{H} & \mathbb{H} \end{tikzcd}$$

with $h\circ i \in \mathbb{H}$ and $(gh)\circ i \in \mathbb{H}$.

Hence the actions of $SL_2(\mathbb{R})$ on $SL_2(\mathbb{R})/K$ and on $\mathbb{H}$ are equivalent.

Rk
$\mathbb{H} = SL_2(\mathbb{R})/K$ is a so called Homogeneous space

The Lie gp. $G = SL_2(\mathbb{R})$ acts on $X = \mathbb{H}$ transitively. $\mathbb{H}$ is homogeneous in the sense that under the action of G, $\mathbb{H}$ looks locally the same.
(A smooth manifold M endowed with a transitive smooth action by a Lie group is called a Homog. space).